x2-05-09 - A-Logic and Computer Technology --- R. B. Angell, May 9, 2002

This paper is intended to suggest reasons why computer manufacturers and programmers might be interested in examining A-logic with an eye to its implications for computer technology.

A. What is A-Logic? A-logic is an alternative to mathematical logic. It is more useful than mathematical logic because 1) although it contains all theorems of mathematical logic 2) it avoids logical paradoxes and anomalies with respect to ordinary usage that infect mathematical logic, 3) it provides accounts of logical procedures in common sense and empirical science that mathematical logicians would like to have accounted for but found that mathematical logic could not provide.

By "mathematical logic" I mean the first two parts (propositional logic and quantification theory) of the logic (without identity or set theory) initiated by Frege in 1879 in his *Begriffschrift* and developed later by Bertrand Russell and Alfred North Whitehead in *Principia Mathematica*, v.1, Cambridge, 1910, and all of its versions in hundreds of textbooks in schools and universities throughout the world. Among logicians and mathematicians this logic is commonly called "standard logic" or "classical logic" or "symbolic Logic" as well as "mathematical logic". Since there can be other logics of mathematics, it would be best to designate this logic by a more neutral term . Below I use 'M-logic' for this logic developed by Frege, Russell and Whitehead and their successors, though elsewhere I use the less careful term "mathematical logic".

M-logic has been enormously more powerful and rigorous than any previous system of logic. It has infinitely more forms of valid argument and has provided proofs for the logical validity of mathematical proofs that no previous system could provide. The computer technology that dominates the world today is based on central concepts of M-logic (e.g., digital computers use electrical circuits based on mathematical logic's truth-tables for 'and', 'or' and 'not').

But despite its great positive achievements M-logic has been bedeviled by problems. A-logic is the result of attempts to solve these problems while retaining all the substantial achievements of mathematical logic. The problems in M-logic are well known by its leading proponents and supporters but usually are not considered important enough to negate M-logic's great advantages.

How A-logic differs from M-logic. The central concept in formal A-logic is that of logical synonymy -- logical sameness of meaning. This concept enters into the definitions of all logical relations and properties in A-logic. It is central to each new development in the book, chapter by chapter. All axioms are statements asserting logical synonymy, and all theorems are based on determinations of logical synonymy or its derivative, logical containment. In contrast, M-logic is based on the concept of truth or falsity and truth-functions.

In A-logic the word 'synonymy' is given a meaning rigorously correlated with purely syntactical properties and relations of expressions whose meanings stand in that relation.

A definition of "synonymy" must satisfy three necessary conditions: 1) both expressions must refer to or talk about all and only the same entities, 2) both expressions must say or predicate the same things about each of those entities, and 3) all and only what is logically implied by one must be logically implied by the other. Put negatively, if two sentences do not refer to (talk about) the same individuals, or do not say the same things about each of those individuals, or have different logical implications they can not have exactly the same meanings.

In A-logic the concept of *logical synonymy* is correlated with the following set of syntactical criteria symbolized by 'P SYN Q'. Two well-formed-formulae of formal logic are SYN if and only if joint instantiation of the two wffs will always contain all and only the same elementary wffs, and whatever is logically contained in one is contained in the other. Thus to be a pair of synonymous expressions, two wffs must have 1) all and only the same letters for individual constants, and 2) all and only the same predicate letters applied to the same individual constants. (Otherwise instantiated sentences could refer to different entities and/or say different things about them) and 3) In addition, they must *logically contain* the same expressions. The six axioms of formal axiomatic logic satisfy these conditions. They consist of synonymous pairs of wffs expressing the familiar principles of idempotence, commutation, association, distribution, double negation, and A-logic's interpretation of *modus ponens*. The rule of SynSUB (Substitution of Synonyms) is used in lieu of the truth-functional version of modus ponens throughout the system.

The notion of *logical containment* is central in A-logic. It is correlated with the syntactical relation CONT. 'P CONT Q' (for "P logically contains Q") is defined as "P is logically synonymous with (P&Q)". In symbols: 'P CONT Q' $=_{df}$ 'P SYN (P&Q)'. For example, putting 'R&Q' for 'P' in that definition, '(R&Q) CONT Q' $=_{df}$ '(R&Q) SYN '((R&Q) & Q)'. And since (R&Q) ***is*** synonymous by A-logic's axioms with '((R&Q) & Q)', it follows that [(R&Q) CONT Q] is a theorem of A-logic. Logical containment, it turns out, is like Simplification writ large; P logically contains Q if and only if a <u>synonymous</u> basic conjunctive normal form of P has Q as a conjunct. In sentential logic this is decidable syntactically.

Logical synonymy is a stronger relation than the 'logical equivalence' in M-logic (which is established by sameness of truth-tables). Sameness of truth-tables is a necessary, but not sufficient, condition for logical synonymy. P and (P & (PvQ)) have the same truth tables but are not synonymous because they don't have the same letters. ((P&~P) & (Qv~Q)) and ((Pv~P) & Q&~Q)) have all of the same letters, and the same set of elementary wffs, but the first logically contains (P&~P) while the second does not. The syntactical relation of 'SYN' has always held for an important sub-class of pairs of M-logic's well-formed formulae but it has never previously been recognized.

B . Problems of M-logic and how A-logic solves them. There are two main sources of M-logic's problems: 1) its definition of 'validity' and 2) its interpretation of 'if...then'. Among problems due to its definition of validity are the following two.

1. **The "paradoxes" of strict implication.** These are not strictly logical paradoxes, but are certainly anomalous. According to M-logic's definition of "validity" it follows (contrary to educated common sense) that 1) an argument is valid no matter what the premisses may be, if the conclusion is tautologous (i.e., the denial of an inconsistency) and also that 2) if the premisses are inconsistent, every statement whatever follows validly from those premisses. Educated common sense judges arguments of these sorts *non sequiturs* -- the conclusion does not follow from the premisses. It is the definition of 'validity' in M-logic (as the concept of "its being impossible that the premisses are true and the conclusion false") that makes these arguments all "valid" according to its definition. For if the premisses are inconsistent, they can't be true, and if the consequent is logically true it can't be false; thus in neither of these cases can the premisses be true and the conclusion false..

[<u>Note</u>: it is *arguments* of this sort that A-logic labels *non sequiturs*; not the truth-functional "theorems" that M-logic correlates with such arguments. The term 'paradoxes of strict implication' has been applied erroneously to the related "theorems". By M-logic's meaning of 'if...then' "if (P & not-P) then Q" is a tautology (hence a "theorem" of M-logic), since it is synonymous in M-logic to the denial of an inconsistent statement, i.e., to "~(P & ~P & ~Q))". This "theorem" is anomalous only if expressed with M-logic's 'if...then' and called "valid". That it is tautologous and unfalsifiable is undeniable. But *arguments* related to this "conditional" by having its antecedent for premiss and its consequent as conclusion, are *non sequiturs* and invalid in every ordinary sense.]

2. **The Liar Paradox.** M-logic is based on a semantical theory in which the "logical validity" or "theoremhood" of a statement depends on the impossibility of its being false. In 1944 Tarski argued that M-logic can not deal with statements that assert a sentence is true or false, because this would lead to the Liar Paradox.[1] By M-logic's rules for "valid" inference, its axioms and definitions, if we have statement named S which states that "S is not true" then a contradiction,

'S is true & S is not-true', follows validly from S. But in all theories of logic an inference can not be valid if it proceeds from a possibly true premiss such as S to a contradiction. That the inference from S to a contradiction is derived by M-logic's rules of "valid inference", yet leads from a possibly true premiss to a contradiction is what makes it a logical paradox.

Both of these two problems are avoided in A-logic by its definition of 'validity'. In A-logic only arguments (inferences) and conditional statements can be valid. In A-logic an argument (or conditional) is *logically valid* if and only if (a) the premisses (antecedent) logically contains the conclusion (consequent), and (b) the conjunction of premisses and conclusion (antecedent and consequent) is not inconsistent. The first set of problems above is removed by clause (a), the second problem is avoided by clause (b) in A-logic's definition of 'validity'.

The second type of problem -- resulting from M-logic's interpretation of 'if...then' -- are due to M-logic's view that "if p then q" means the same as "it is not the case that both p and not-q" or equivalently , "either not P or Q". It follows from these that (a) if the antecedent is not true, then every conditional with that antecedent (no matter what the consequent may be) is true, and (b) if a consequent is true, then every conditional with that consequent (no matter what its antecedent may be) is true. These and many related consequences were initially attacked as "paradoxes of material implication". But strictly speaking they are not logical paradoxes. Rather, they are anomalies due to the un-ordinary meaning given to "if...then". They are better called "anomalies of the truth-functional conditional". However, this mis-interpretation of 'if... then' puts obstacles in the way of explaining important logical procedures used in common sense and the empirical sciences. Among this group of consequences are the following

3. Problems of confirmation, dispositional predicates and counterfactual conditionals. Carnap, Hempel and Goodman were staunch proponents of mathematical logic. But in trying to extend it to the empirical sciences in 1938 Carnap found that M-logic can not be used to define important dispositional predicates [2]; if "a is soluble" were defined as "If a is put in water, then a dissolves", then with M-logic's 'if...then', it would follow that all things that are never put in water (e.g., all the rocks on Mars) are soluble. In 1945 C. G. Hempel wrote that, on the basis of M-logic "we have to recognize...[that] any green leaf becomes confirming evidence for the hypothesis that all ravens are black"; for, since a green leaf is not a raven, it follows that if x is a green leaf then "If x is a raven then x is black" is true.[3] In 1947 Nelson Goodman pointed out that if contrary-to-fact and subjunctive conditionals were taken to be conditionals in M-logic's sense, then the truth of "If that piece of butter had been heated to 150E F, then it would not have melted" would be logically implied by the fact that that piece of butter (e.g., some butter I eat) was never heated to 150E F.[4] All of these unwanted consequences follow from the fact that when 'if...then' is interpreted as M-logic interprets it, 'if...then' statements are true whenever the antecedent is false.

4. The problem of conditional probability. In 1965 Ernest Adams showed that the probability that Q is the case if P is the case, is not the same (with the "if....then" of M-logic) as the conditional probability of 'Q, if P' in standard probability theory.[5] If there are five apples one of which is green, and five green pears in a basket, the chances that a piece of fruit in that basket "is green, if it is an apple", is 0.2 according to probability theory, but the probability that it "is green, if it is an apple" with M-logic's "if...then", is the probability of "Either it is not an apple, or green" which is 0.6. The logical analysis of probability theory can not be carried through using the 'if...then' of M-logic to get conditional probability.

In A-logic problems 3 and 4 are solved. According to A-logic's interpretation of "if...then" a conditional is neither true nor false if the antecedent is not true. To prove a conditional statement true or false, the antecedent must be true. The obstacles in problem 3 are thereby removed; in A-logic no conditional is true just because its antecedent is always false. Subjunctive conditionals in which the antecedent is not fulfilled can not be proven true in A-logic, but they can be proven logically valid if they are consistent and the meaning of the consequent is contained in the meaning of the antecedent. Logically valid conditionals in A-logic can not be false, but they need not be true. Further, using A-logic's 'if...then' instead of M-logic's, the probability of A-logic's conditional turns out to be the same as the concept of conditional probability in standard probability theory.

A-logic succeeds where M-logic fails. It explains the logical nature of confirmation and of universal law-like statements in empirical science and common sense -- including laws involving dispositional properties, contrary to fact conditionals, and probabilities -- despite the fact that universal law-like assertions of fact can not be conclusively proven true.

C. Consequences of A-logic significant for computer science include the following:

a) When it comes to questions about the truth of sentences, A-logic needs three "values" namely: truth, falsehood and neither-true-nor-false. This is because it holds that statements of the form 'if P.then Q' are (1) **true** if and only if both antecedent and consequent are true, (2) **false** if and only if the antecedent is true and consequent false, and (3) **neither-true-nor-false** if the antecedent is not-true or the consequent is neither-true-nor-false. This differs from M-logic where. every statement is either true or false exclusively, and "if P then Q" is said to be true if either the antecedent is false or the consequent is true. A-logic's three-valued version of "if P then Q" is basic.

Can computers be based on a ternary, rather than a binary system? Can computer chips be designed so that a byte is able to be in any of three distinct states? What advantages, if any, would there be in having trivalent bytes?

b) In A-logic trivalent truth-tables are provided for (I) the primitive connectives 'and' and 'not' (and based on these for the exclusive 'or' and the truth-functional "if...then" symbolized by 'Pe Q') and also for (ii) the primitive operator 'It is true that...' (and with 'not', for the derived operators 'It is false that...' and 'It is neither-true-nor-false that...'), and for (iii) the primitive non-truth-functional C-conditional "if...then", symbolized by "P ⇒ Q". Where there are definite truth-tables electrical circuits can mirror them.

c) The rules governing the trivalent truth-tables are all expressible in A-logic, and are proven to be logically valid. (See Tables on pp 456 and 457 and related text). Analogous rules are not expressible within M-logic). The set of trivalent truth-tables is truth-functionally complete.

d) The presence or absence of the basic semantic relations of A-logic -- logical synonymy and logical containment -- as well as the property of logical inconsistency, are computable. They are determined by the presence and positions of two or more occurrences of the same variable (or set of variables) within a pair of logical formulas or within the antecedent and consequent of single conditional formula. Axioms, definitions and rules of inference govern these determinations.

e) The primary vehicle in derivations is the rule of inference SynSUB, the substitution of synonyms. It moves faster than *modus ponens*. Routines for reducing to "basic conjunctive normal forms" give quick determinations of logical containment. Computers can handle this.

f) Thus the applicability of the predicates '...is logically synonymous with...', and '...logically contains...', to any pair of well-formed logical formulae is computable. Within specifiable range 'is inconsistent' is also computable.

g) The "logical validity" of inferences and conditionals is defined by logical containment and inconsistency: The logical validity or invalidity of any formula is computable except for certain classes of quantified formulae that are undecidable with respect to inconsistency.

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1 Alfred Tarski, "The Semantic Conception of Truth", *Philosophy and Phenomenological Research*, 1944

2 Rudolf Carnap, "Testability and Meaning" (Section II, 4, Definitions) *Philosophy of Science*, 3, 1936,

3 Hempel,Carl .G.,"Studies in the Logic of Confirmation", *Mind*, v.54, 1945 , pp 1-26
4 Goodman, Nelson, "The Problem of Counterfactual Conditionals", *Journal of Philosophy*, v.44, 1947
5 Adams, Ernest, "The Logic of Conditionals", *Inquiry*, Vol 8, pp 166-97, 1965

A-LOGIC

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** Solves Raven Paradox, Paradox of Confirmation, Problems of Dispositional Predicates, Contrary-to-fact or Subjunctive Conditionals, Formulation of Causal Statements, Conditional Probability and the Probability of a Conditional**.**