The Geometry of Visibles[1]

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I

The thesis I wish to defend is this: the geometry which precisely and naturally fits the *actual* configurations of the visual field is a non-Euclidean, two-dimensional, elliptical geometry. In substance, this thesis was advanced by Thomas Reid in 1764, although, since this preceded non-Euclidean geometries by 65 to 90 years, no mention was made of them.[2] The thesis conflicts now, as in Reid's time, with the view held by most psychologists and philosophers that the actual two-dimensional configurations of the visual field are Euclidean. It also appears to conflict with the recent theory of Luneburg [8] that the geometry of binocular space perception is, though non-Euclidean, hyperbolic rather than elliptical; but we shall see that this latter difference is merely apparent—Luneburg's theory deals with a different problem.

To grasp the import of our thesis it is important to distinguish at the outset:

(*Ap*) *Actual* geometrical properties of and relations between *physical objects.*

(*Jp*) *Judgments or perceptions* of geometrical properties of and relations between *physical objects.*

(*Av*) *Actual* geometrical properties of and relations between *visual objects* (visibles).

(*Jv*) *Judgments or perceptions* of geometrical properties of and relations between *visual objects* (visibles).

Common sense and natural science agree that *Ap,* the actual geometrical properties and relations of physical objects, satisfy the axioms and theorems of three-dimensional Euclidean geometry—at least in the measurable range between the astro-

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nomically large and the sub-microscopically small. With this we have no quarrel at all. When psychologists study perceptions or judgments of the geometrical relations and properties of physical objects, *Jp*, they usually refer to and describe (among other things) the visual cues, i.e., what they take to be actual geometrical properties of visibles (*Av*), which give rise to these perceptions or judgments. Thus, the psychologist will tell us that the perception, *Jp*, of a table as having a square top—or the judgment that there is a table before me which has a square top—is normally based on certain *Av*, actual geometrical properties of visibles, e.g., one or more configurations in my visual field which are actually trapezoids, along with other psychological or physiological mechanisms. Sometimes, of course, psychologically normal perceptions or judgments about three-dimensional physical objects are non-veridical or illusory; this is the case if and only if the geometrical properties perceived or ascribed in *Jp* do not coincide with the actual geometrical properties of the physical objects referred to, *Ap*, as determined by measuring the latter with measuring sticks, etc. In general, psychologists are interested in the mechanisms or normal processes by which perceptions of Euclidean three-space, whether veridical or not, are formed. Thus, we shall see that Luneburg's theory of the geometry of binocular space perception is in fact a theory based on certain relations between *Jp* and *Ap*, i.e., between what men normally *perceive* as being parallel or equidistant among physical objects, and the actual geometrical relations of the physical objects involved. With these findings, too, we have no particular quarrel; but it is important to see that the subject matter of Luneburg's thesis is different from the subject matter of our thesis.

To grasp the import of our thesis, it must be recognized that *Jv*, judgments or perceptions of geometrical relations and properties of *visual* objects, may also be non-veridical or illusory as well as veridical, i.e., that these judgments or perceptions may or may not diverge from *Av*, the *actual* geometrical relations and properties of the visual objects involved. Thus, we do not deny, for example, that most men, including most psychologists and philosophers, now *think* or *judge* or *perceive* the trapezoidal appearance (the visual object) they experience when looking at a square-topped table (the physical object) from one side as being a *Euclidean* trapezoid, i.e., a trapezoid with interior angles adding up to four right angles. But our thesis is that *all* such judgments or perceptions, *Jv*, which ascribe Euclidean properties to visual objects are non-veridical. The actual geometrical properties and relations of visual objects, *Av*, are *not* Euclidean, we claim. Despite the fact that it is so widely thought that they are Euclidean, we propose to show that actual geometrical relations of visibles, *Av*, satisfy just the axioms and theorems of a two-dimensional, bipolar, elliptical geometry. Thus, for example, the actual trapezoids found in our visual field can be shown to have interior angles which measurably and significantly add up to more than four right angles, though never more than eight, as in elliptical geometry.

If our thesis is correct, a great many intelligent men have been systematically mistaken for many centuries. This impels us to try to account for the occurrence of such widespread systematic error, in addition to giving direct arguments in favor of our thesis. We must also give some attention to the objective basis for distinguishing the "actual" geometrical properties of *visibles* (*Av*) from the "judged or perceived" geometrical properties of *visibles* (*Jv*): some analogue, perhaps, to the measuring methods and devices for physical distances which render perceptions of such distances corrigible. We shall try to meet both these requirements. In the next section, we outline the direct evidence in favor of our thesis, with special attention to methods and devices for objective measurement of distances among actual visibles. In the third section, we examine the relationship between the Luneburg theory and our own. In the final section, we deal with questions stimulated by the radical divergence between our thesis and the more commonly accepted judgments of intelligent men.

II

The term 'a visible', used as Thomas Reid used it ([10]), refers to a kind of object which any normal human being can be aware of, attend to, and describe fairly accurately when his eyes are open—or more exactly when one of his eyes is open, for we shall begin by considering only monocular vision. Visibles or visual objects are not the same as what would ordinarily be denoted by 'an object which is visible'. Thus, I might say that a certain tree was an object which was visible to me at a certain time; the "object which is visible" in this case is a physical, three-dimensional object, a tree. I might say of that tree that I judged it to be about seventy feet tall; this, a judgment of a geometrical ratio of two physical Euclidean objects (a foot rule and the tree), is a *Jp*, in the language of the preceding section. But I might also say that the tree *appears* to be no larger than my thumb *appears* when held at arm's length. The *appearances* thus compared and found equal are the visual objects, or visibles, in our present sense. They are the sorts of entities which a landscape painter must attend to—eliminating his perceptual presuppositions of three-dimensional sizes and shapes—in order to dispose shapes and colors on a flat two-dimensional canvas so that the latter will "look just like" the real physical things "look". Traditional laws of perspective for draftsmen are useful to the painter or draftsman. To insure that the visibles an observer will experience when standing in front of a flat piece of drawing paper have the same actual visual shapes (*Av*) as the visibles he would experience when looking at the given physical landscape, the draftsman must draw two-dimensional figures on the flat paper whose comparative sizes, shapes, and geometrical relations are very different from the comparative sizes, shapes, and geometrical relations which obtain in the actual physical landscape (*Ap*), and the former must be related to the latter by the strict rules of perspective. We must not, however, commit the common error of confusing perspective geometry or the two-dimensional Euclidean geometry of the flat Euclidean plane (the paper) in three-space, which contains the draftsman's drawings, with the two-dimensional geometry of the *visibles* which are experienced by *looking at* the flat drawing or by *looking at* the real landscape from the selected standpoints. Visibles are neither the physical configurations on the flat paper, nor the physical configurations of the landscape which is pictured; both of the latter conform to Euclidean Geometry, but the configurations of visibles do not. Were we physicalists, we might try to *explain* the geometry of visibles in terms of the geometry of impressions on the retina of the eyeball (spherical geometry). But the purely phenomenological account of the geometry of visibles given here in no way depends upon such an explanation: the non-Euclidean properties of actual visibles are determined in a natural way which is quite independent of any such physical, or metaphysical, explanations.

The visibles found in the pure visual field will include points, lines, and areas or regions. But what determines which geometry is satisfied in the actual visual field are determinations (i) of actual visual distances between points in this field and (ii) of actual sizes of angles among points or staight lines in this field. By "visual distance" we obviously do not mean the psychologist's "depth perception"; the third dimension, or depth along the line of sight, has no place in the two-dimensional geometry of visibles. Rather we mean the directly sensed or seen distances within or among visual appearances. The reader can readily determine if he holds one forefinger six inches in front of his eye and the other forefinger at arm's length that the *visual appearance* of the former is about three times the size, in width or length, of the *visual appearance* of the latter. As visual objects, or visibles, such appearances exist side by side in the same two-dimensional field; the third dimension, or distance along the line of sight, is irrelevant, and not to be found in this field. Again, the size of angles among visibles can be apprehended and described fairly accurately by direct inspection; though a physical table may be known to be square-topped, we can easily determine that the *appearance,* or *visible,* of that top contains only obtuse and acute angles.

The term 'visual field' refers to the two-dimensional continuum which contains visibles. Intuitively, the momentary visual field of a given observer is the two-dimensional expanse of visual colors and shapes which the observer can be aware of at that moment. In fact, of course, a normal person's momentary visual field is limited by boundaries determined by the nose, eyebrows, etc. and never includes more than perhaps one quarter of what it could contain if a person could see in all directions at once. But we shall also want to speak, eventually, of motions and changes as well as fixity among the visual objects in the visual field. We must therefore think of the visual field of a given person as persisting through time and containing, or having as members, a series of the observer's momentary visual fields. But, also, we shall want to speak of a given person's visual field as more extended spatially than any momentary visual field is in fact. In following a line with the eye (e.g., scanning the horizon), we take portions of the line previously scanned but no longer in the momentary field to be continuous with the portions later scanned. We will therefore speak of a person's total visual field as that expanse which includes all possible continuous extensions of lines or regions in his momentary visual field. This may be interpreted as including all that such a person could be visibly aware of if he could see in all directions at once at a given moment, as well as changes among visibles which could be noted over a period of time. Thus, though only one's momentary visual field is directly and immediately observable at a given moment, the concept of a person's total visual field is a construction based on relationships found within and between momentary visual fields.

Now, the universally accepted thesis that the geometrical configurations of actual (macroscopic) *physical* objects (*Ap*) satisfies the axioms and theorems of a three-dimensional Euclidean geometry is given strong supplementary support by the availability of measuring instruments like meter sticks for physical distance and protractors for physical angles. Had we no such instruments, the universal applicability of such Euclidean theorems as the Pythagorean theorem or the theorem that the sum of the interior angles of a triangle always equals two right angles would rest upon our judgments or perceptions of physical configurations (*Jp*) (supplemented, perhaps, by such relatively crude methods of measurement as using one's hands or feet as units of measurement). With more refined instruments, we can make more precise measurements and prove, to almost any desired degree of accuracy, the accuracy of the Euclidean theorems with respect to actual physical objects. Judgments and perceptions (*Jp*) thereby become corrigible by objective measuring devices of *Ap*. The correctness of our claim that the geometry of visibles is in fact a non-Euclidean, elliptical geometry gains an analogous objective support by the availability of instruments which, though less familiar, can measure the actual distances and angles found in the visual field. Our case does not rest, basically, upon the presence of such instruments; the main evidence which we present below is evidence available by commonplace direct experience of visibles unaided by instruments. Such instruments will, however, allow us to confirm the theorems of non-Euclidean geometry for those cases where men think or judge or perceive the geometry of visibles (*Jv*) to be otherwise. They provide for the objective corrigibility of judgments about the geometrical relations among visibles.

The instruments available for visual geometry differ from those for physical objects. One method of measuring visual distance more accurately is related to methods astronomers use to measure the "angular distance" between stars. Ordinarily, when we think of the astronomer as measuring angular distance, we think of observing him from above (in three-space), and we think of the angle between the imaginary straight lines drawn from the two stars to the astronomer's eye; this angle is sometimes called the "visual angle", and is expressed in degrees: 3°, 90°, 153°. Essentially this same method may be applied to the *appearance* (visible) of my forefinger; such an appearance has visual length of 30° if my finger is six inches from my eye, and the appearance (visible) when my forefinger is at arm's length is about 10°. However, the concept of observing the astronomer from above as an object in three-space is misleading. For what a man *sees* when he *sees* a visual distance between two visibles is not an angle between two lines seen from above; he simply sees a visual expanse in the visual field between two visible points of visual space. One simple instrument which will measure visual distances in this sense objectively can be constructed as follows. Take a stick 14.35 inches long, attach a six-inch metal strip marked off in quarter inches to one end of it, and bend the metal strip so that each point on it is equidistant from the free end of the stick. When the free end is placed just below the eye, the quarter-inch marks on the metal strip at the other end each mark off just one degree (or 1/360th of a complete horizon) of visual distance. A cruder instrument is one's own fist, which when held at arm's length from the eye produces a visible about 8° or 10° of visual width, with about 2° to 2.5° per finger. More precise instruments would be variations on the surveyor's theodolite or the ship-captain's sextant. We shall see later that the visual field determines (unlike the field of physical objects) certain absolute units of measurement for visual size.

For objective measurement of seen *angles* among visibles (e.g., for measuring the actual size of visible angles in the trapezoidal visible which is produced by the square table-top when viewed from one side), it suffices to attach a protractor perpendicularly to the same stick, with its center at the end where the metal strip is attached. When this device is held to the eye and the angles in the protractor are aligned with the angles in the visible, an accurate, objective measure of the angles in the visible is provided.

The fact that the instruments we have just described are physical objects, describable in Euclidean terms, should not mislead us into supposing that the geometry of visibles is in any way essentially dependent upon or necessarily a derivation from Euclidean geometry. What is being measured still remains two-dimensional visibles, not three-dimensional physical objects. And, further, it would be possible, with a complicated analysis not possible in this paper, to construe all three-dimensional objects, including these instruments, as constructions derivable from our non-Euclidean geometry of visibles taken as a fundamental point of departure.

Starting with these metrical concepts of visual distances and angles associated with distinct operations and instruments, it is possible to give practically effective and rigorous definitions of visual circles and visual straight lines in the visual field:

> A *circle* is a closed line such that all points on it are equidistant from a point
>
> A line segment *AB* is *straight* if and only if the distance from end-point *A* to end-point *B* is equal, for each point *C* on *AB*, to the distance from *A* to *C* *plus* the distance from *C* to *B*.

From these definitions, we may proceed to define angles, specifically, right angles; bisection of angles; then triangles, quadrilaterals (closed figures with four straight lines as sides), rectangles (equiangular quadrilaterals), squares (equilateral rectangles); and so on. Note that the definitions given would all be equally suitable for plane or solid Euclidean geometry, as long as the operational metric for "distance" is left indeterminate.

Now the question arises whether the geometrical propositions which hold of the objects of the pure visual field belong to Euclidean geometry or not. The answer is plainly that they do not. Two-dimensional Euclidean geometry includes such theorems as the following:

1. A straight line cannot be a circle.
2. Every straight line is infinitely extendable.
3. Two straight lines intersect at most in one point.
4. Two straight lines, cut by a third straight line perpendicular to both, never intersect.
5. All equilateral triangles have the same interior angles.
6. The sum of the interior angles of a triangle equals two right angles.
7. The four angles of a rectangle are all right angles.

By precise measurements of visual distance, it is shown that none of these theorems holds for the straight lines, triangles, and rectangles of the pure visual field. On the contrary,

1′. *A straight line can be a circle;* i.e., a visual straight line can be a closed line with all points on it equidistant from a polar point in the visual field. Consider the horizon, with a point directly overhead as its center.
2′. *No straight line is infinitely extendable.* If we extend any straight line segment in the visual field, it eventually returns on itself. It is thus finite, though unbounded. Again, consider the horizon.
3′. *Every pair of straight lines intersects at two points.* Imagine standing in the middle of a straight railroad track on a vast plane. The *visual* lines associated with the two rails are demonstrably visually straight in every segment—they *appear* perfectly straight, not curved, visually. Yet these visually straight lines meet at two points which are opposite each other on the horizon, and they enclose a substantial region on the visual field.
4′. *Two straight lines, cut by a third straight line perpendicular to both, always intersect.* The two rails, both appearing visually straight, are cut by the straight edge of the railroad tie at our feet, and this tie is perpendicular, visually, to both of them; yet the two visual rails intersect twice.
5′. *All equilateral triangles do not have the same interior angles.* Consider a large visual triangle, like that between a star due east on the horizon, a star due north on the horizon, and a star directly overhead. In this case, equal visual straight lines connect the three stars, so the triangle is equilateral. Yet the angles are all right angles and thus are larger than angles of smaller equilateral triangles which approach 60°.
6′. *The sum of the interior angles of a triangle is always greater than two right angles.* This was the case in the visual triangle described above and would be found to be the case for all other visual triangles upon careful measurement.
7′. *The four angles of a rectangle are always larger than right angles.* This is clear if we measure the visual angles—that is, the angles appearing in the visual field—of, say, a

> picture frame if it is visually rectangular. It is not necessary to use instruments to see this. We can approach a picture frame so that the sides are not only all straight, but the angles are all visually equal. Yet the equal angles are all obtuse, visually.

Now, all of the counter-Euclidean propositions just enumerated are theorems in the two-dimensional, non-Euclidean, bipolar, elliptic geometry of Riemann. In fact, all theorems of that geometry will fit precisely (so far as precision is possible) the configurations of the visual field. To those familiar with this type of non-Euclidean geometry, the remarks above should be sufficient proof that it describes rigorously the geometry of the pure visual field in monocular vision.

Standard expositions of non-Euclidean geometry usually suggest that these geometries are not incompatible with sense experience by two lines of argument. It is pointed out that we might conceivably find, as we made direct measurements of larger and larger objects, that Euclidean laws began to fail, that, for example, sufficiently large triangles had noticeably more than two right angles as the sum of their interior angles. Or, secondly, we are told of Euclidean *models*—hyperbolic, saddle-shaped surfaces, or the surface of a sphere—of which, provided we redefine our terms, the non-Euclidean geometries would hold. The first line of argument does not suggest a natural or common sense application of non-Euclidean geometry to any familiar macroscopic objects; if it holds at all, it must hold of astronomically large portions of the universe. Thus, it suggests a theoretically possible, but not an actual, directly observable, set of empirical data which satisfy the non-Euclidean postulates. The second suggestion serves well enough as a consistency proof, but scarcely suggests a natural or ordinary-language application of non-Euclidean geometry, since the surfaces of spheres and saddles are recognized as belonging to Euclidean geometry, and the model works only by the terminological devices of calling lines which everyone recognizes to be curved lines, "straight lines".

In contrast to these ways of relating elliptical geometry to sense experience, the preceding considerations specify a domain of objects (visibles) which is constantly available to every normal person, where the words "line", "angle", "straight", "circle", "rectangle", "triangle", etc. retain established, ordinary uses and where, even without instruments, we can see that the theorems of elliptical geometry are immediately and incontrovertibly applicable. Any normal man can distinquish very well between visibles which are straight lines (qua visibles) and those which are curved. He can agree that the appearances produced by the two railroad tracks are both straight lines and yet intersect at two points. And he can tell with fair accuracy, without instruments, whether a visual object is a square, a rectangle, a circle, a trapezoid, etc. Our definitions are thus completely compatible with ordinary language as it is used with respect to visibles, or visual appearances. Further, though men's judgments or perceptions of those visibles may diverge or go astray for psychological or other reasons, we have, with the instruments mentioned, methods for rendering such non-veridical judgments corrigible on an objective basis.

Finally, we may point out as promised, that, in keeping with the theorems of elliptical geometry, absolute units of visual distance are definable either in terms of (1) the length of sides of a triangle with three right angles (a quadrantal triangle), which are always 90° of visual distance, or in terms of (2) the distance between the two points of intersection of any two straight lines, which is always 180° of visual distance. Letting either of these units serve as distance 1, we can, by bisection, set up a system of measurement (more convenient than 360° measure) to any theoretically desired degree of accuracy.

III

In R. K. Luneburg's *Mathematical Analysis of Binocular Vision* [8], it is also held, on the basis of empirical findings, that "binocular visual space" is non-Euclidean. But, for Luneburg, it is the hyperbolic geometry of Bolyai and Lobachevski, not the elliptical geometry of Riemann, which characterizes binocular visual space. Thus, his conclusions may appear to be even more at variance with the thesis of this paper than the common view. For in hyperbolic geometry,

1. Through a given point not on a given line more than one line can be drawn not intersecting the given line.
2. The interior angles of a triangle total less than 180°.
3. The interior angles of a quadrilateral total less than 360°.

4. Two non-intersecting lines continuously diverge on each side of their common perpendicular. (Cf. [12].)

An examination of the experiments which form the basis of Luneburg's conclusions show, however, that there is no conflict.

Luneburg's theory is based upon a type of experiment first investigated By W. Blumenfeld in 1913. (See [4].) In a darkened room, the subject sits at a table with his head held in position so that only his eyes can move. Two rows of tiny starlike lights are placed on either side of the median (the vertical plane in the center of his line of sight as he looks straight ahead). The two farthest lights are fixed symmetrically and equidistant from the median. The subject is asked to do two things. First, he is asked to arrange the other lights so they will form a "parallel alley" extending towards him from the fixed lights. That is, he is to arrange the two lines of lights on either side of the center so that he *judges or perceives* them as being straight and parallel to each other in three-dimensional space. The second task is to construct a "distance alley". For this experiment, all lights except the two fixed lights and two others are turned off. The subject must arrange the two lights so that they are perceived as nearer to him but as being the same physical distance apart as the fixed lights, and physically equidistant from the median. Then the last two lights are turned off and the task is repeated with another pair of lights closer to him, and so on. The *actual physical arrangements* of lights after they have been arranged in each of the two tasks are recorded, and the results, typically, are of the form shown in *Figure 1:* the *actual* physical configuration of lights which the subject judged to be "parallel" alleys diverge (rather than being actually parallel) and are set nearer to the median or center line than the actual lines of lights which he *judged* to be equidistant. From the fact of the divergence between actual physical objects aligned under these conditions, Luneburg infers that the *perceptions or judgments* of parallelism and of equidistance in binocular vision do not coincide. Now, in Euclidean geometry, of course, two straight lines are parallel if and only if they are also equidistant from each other at all points. Among non-Euclidean geometries, two straight lines equidistant from each point on a median either converge (in elliptical geometry) or diverge (in hyperbolic geometry). Since

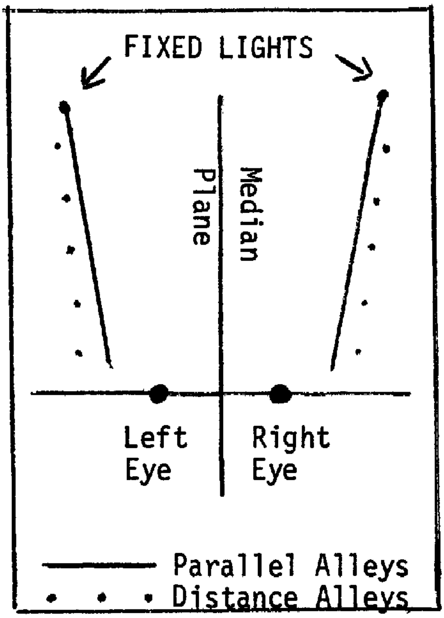

Figure 1

the experiments show a statistically strong tendency towards divergence, with a greater tendency to divergence in parallel alleys than in equidistant alleys, Luneburg concluded that the geometry of binocular *perceptions* of the spatial relations of physical objects satisfies the fourth theorem above of hyperbolic geometry. Other experiments along these lines satisfied other theorems of hyperbolic geometry and trigonometry in the same domain.

Neither the accuracy of these experimental results, nor their connection with hyperbolic geometry, need be questioned here. The first point is that they deal with a different domain of data than that of our thesis. The domain within which these hyperbolic properties and relations are alleged to exist is the field of *Jp*, of judgments about, or perceptions of, geometrical relations and properties among physical objects. The domain which *we* assert satisfies the axioms of elliptical geometry is the domain of *Av*, the actual geometrical relations and properties which are found among visibles. The data on which Luneburg bases his theory are *Jp*, reported judgments or perceptions about three-spatial relations from subjects, and *Ap*, actual physical measurements of actual physical objects. The data we appeal to for our thesis are *Av*, actual visual distances and angles as directly observed or measured by instruments mentioned above. Even if we grant the correctness of Luneburg's conclusions, there is no conflict between the two theories.

Our second point is that it seems quite possible that Luneburg's results can best be explained by reference to the actual elliptical character of the geometry of visibles (our thesis) together with man's general predisposition towards Euclidean geometry. Consider first the geometry of the visibles presented to an observer when he looks down a pair of straight railroad tracks (which yield straight converging visual lines) with cross-ties. According to our theory, the interior angles of the *visibles* produced by intersections of cross-ties with tracks are not actually equal, as in Euclidean geometry, but successively more acute above and more obtuse below as the lines (from the railroad tracks) converge to a point. The cross-ties intersecting the tracks at the subject's feet yield visual lines which intersect the visual lines from the tracks at right angles, but the angles among visibles produced by such intersections farther away from the subject are successively more acute above and more obtuse below. Consider next what happens if a subject is asked to arrange a string of lights so that they appear, or so that he judges them to be, parallel. If, starting from a fixed pair, he proceeds on the assumption drawn from Euclidean principles of perspective geometry that the interior angles determined by points (visibles) at similar distances on the two lines from the visibles from the two fixed points should all be equal, then his method will lead him to arrange the physical objects in ways which, though satisfying this assumption, will either not produce physically straight, or not produce physically parallel, lines of physical lights. It is quite possible that such efforts would lead to, and account for, just the sorts of results which Blumenfeld found. Now this, of course, is merely a suggestion. The psychology of perception is an experimental science, and no such suggestions should be accepted *a priori;* furthermore, the design of experiments which would test such a conjecture requires more investigation. Nevertheless, it appears at least plausible that the elliptical geometry of visibles might provide an explanation of the hyperbolic characteristics which Luneburg attributes to three-dimensional spatial perceptions on the basis of such experiments.

Thirdly, I wish to remark that a certain amount of confusion is engendered by the failure of philosophers and psychologists alike to distinquish *Jp*, perceptions or judgments of geometrical relations and properties among physical objects, from *Av*, actual geometrical properties and relations of visibles. Thus, A. A. Blank uses the terms 'geometry of vision', 'theory of binocular visual space', and 'binocular space perception' apparently interchangeably in discussing Luneburg's views ([3], [1], [2]).[3] In one of his articles, he writes:

> The ultimate objective of a theory of three-dimensional space perception is to state in some precise way what an observer really "sees" when he looks out at the physical world. ([1]: 717.)

Reid thought that what we "really see" are visibles (*Av*); Blank obviously tends to identify our *perceptions* and judgments of three-space relations (*Jp*) with what we "really see". Common usage, innocent of our distinctions, is no particular help on this. But it seems clear that, one way or another, the distinction *is* a clear and important one, and psychologists and philosophers should see that it is not obscured.

Finally, whatever terminology may be chosen, it seems clear to me that our theory of visibles is one which is supported in the large both for binocular and monocular vision by an enormous array of readily accessible data, while the findings of Luneburg and Blank have a relatively small range of special applications. The geometry of binocular visibles in the open field—i.e., where the visibles are produced by looking at houses, mountains, railroad tracks, and stars—is thoroughly elliptical. In the binocular vision of relatively near objects, double-images constantly appear and disappear among visibles—and on such appearances most of our normal perceptual judgments of physical sizes and distances depends. But all such images, taken separately, satisfy the axioms of elliptical geometry; the occurence of double imagery does not affect this. Since the overlapping of two Euclidean plane figures would not lead to a non-Euclidean composite, why should overlapping non-Euclidean images be supposed to lose any of their non-Euclidean properties?

IV

We thus stand committed in this paper to the view that in some familiar and ordinary senses of 'real' (1) the visibles we have been talking about are real entities, i.e., that some real entities are visibles, and (2) the geometrical properties and relationships among such visibles are *really* those of elliptical, not Euclidean, geometry. We cannot, at this time, give statement (1) the attention it deserves. But if we can, as claimed in (2), assert that certain visibles *really* have, as shown by instruments mentioned above, certain geometrical properties, it would seem we must presuppose some entities, real in some sense, to which such properties belong. Opponents of sense-data theories may see a threat here; I am not sure there is one, but in any case we must pass over their objections in order to get on to the second question, which is our present concern.

Common sense presupposes, as G. E. Moore said in [9], that there have been many, many human beings who have lived and moved about among other physical objects on the surface of the earth in three-dimensional (Euclidean) physical space, and that all of these human beings, save perhaps the blind, have had visual experiences. Given this presupposition, and the history of science (particularly geometry and psychology), educated common sense has a problem of credibility with (2). If, as we said, a visible is "a kind of object which any normal person can be aware of, attend to, and describe fairly accurately when his eyes are open" (p. 89), and yet these objects are such that on careful inspection we find the theorems of elliptical geometry "immediately and incontrovertibly applicable" (p. 97), why has this not been noticed before? And if the thesis is true, how is its truth to be reconciled with the many common sense and scientific principles which presuppose Euclidean geometry? The contemporary unfamiliarity and the historical novelty of our thesis, its concepts, and its operations provide *prima facie* grounds for a healthy common sense skepticism.

Adolf Grünbaum (in [6]), considering Luneburg's non-Euclidean theory, asked a series of questions which should certainly now be asked as well of the present theory. Slightly rephrased to fit our present purposes, these questions are:

(1) How do human beings manage to get about so easily

in a Euclidean physical environment even though the geometry of visual space is presumably elliptical or hyperbolic?

(2) How is man able to arrive at a rather correct apprehension of the Euclidean metric relations of his environment by the use of a physiological instrument whose deliverances are claimed to be non-Euclidean?

(3) How can students be taught *Euclidean geometry* by visual methods, if the geometrical relationships among visibles do not conclude with the Euclidean relationships that are taught by visual methods?

(4) If men have literally been seeing one of the non-Euclidean geometries all along, why did it require two thousand years of research in axiomatics even to *conceive* these geometries, the Euclideanism of physical space being affirmed throughout this period?

(5) Why did such thinkers as Helmholtz and Poincaré first have to retrain their *Anschauung* conceptually in a counterintuitive direction before achieving a ready pictorialization of the hyperbolic (elliptical?) geometry, a feat which very few can duplicate even today?

(6) If we took groups of school children of equal intelligence and without prior formal geometrical education and taught Euclid to one group while teaching elliptical geometry to the other, why is it that, probably, the first group would exhibit a far better mastery of their material?

All of these questions seem to me susceptible to plausible answers from the point of view of educated common sense. Questions (1) and (2) are kinds of questions which psychologists can answer as part of a theory of spatial perception which is not radically different, except in geometry, from currently established theories. Questions (3) and (6) have to do with methods of teaching or learning Euclidean or elliptical geometry and will be seen to pose no great problems when proper distinctions are made. The fifth question can be answered by reference to the history of science, i.e., the manner in which non-Euclidean geometry was introduced, and the fourth question, which I will deal with first, is related to the general question of why many distinctions man makes and *uses* in the pursuit of his ends are not explicitly conceptualized and attended to until man has arrived at a rather later state of science.

Let us consider, then, question (4). Our claim at issue is that all normal men, living and dead, whenever they have had actual visual experiences, have had present to them a field of two-dimensional entities (visibles) and that these entities have had the properties and relations of elliptical geometry. This implies such entities have been *there,* "really present" in some sense, whenever the eyes were open and working, even though usually *not noticed.* It also implies that even when the entities have been noticed and studied—and various philosophers, psychologists, and painters, at least, have studied them and have maintained that they are always capable of being noticed—the true nature of their geometrical properties and relationships have not been (except by Reid) correctly described. Understood in this way, Grünbaum's sentence "men have been literally seeing one of the non-Euclidean geometries all along" does not entail that men have *noticed* or been aware of such non-Euclidean geometries. Rather, it means that there have in fact been present to them, capable of being noticed had they looked properly, certain entities in their field of vision which—had they been carefully enough studied—would have been seen to have the properties and relationships of elliptical geometry and not of Euclidean geometry. Philosophical objections to these entities being "really there but not noticed" will be ignored for the moment; we assume that common sense could treat the claim as meaningful and possibly true. But the fact that no previous thinkers have mentioned these elliptical properties and relations is a good common sense ground for being skeptical of the thesis.

To neutralize this common sense ground of skepticism, we point out first that there are many historical accounts, accepted by common sense, of equally ubiquitous sensible data available to human beings over as long or longer a period of time before being noticed and correctly described by thinkers. For example, the Euclidean principles that the interior angles of a rectangle add up to two right angles and that the ratio of the circumference of a circle to its diameter is π:1 were not noticed and correctly described until some 2200 years ago, although men had been in daily sensible contact with flat rectangular and circular physical objects for tens of thousands of years before that. Again, the distinct conception of uniform accelerated motion, the foundation of modern physics, was not noticed, correctly described, and axiomatized until Galileo's *Dialogue of Two New Sciences* in 1638, although uniform accelerated motion has been there to be noticed every time, in thousands of preceding millenia, a man saw a coconut fall to the ground or threw a stone into the air. In some sense, this concept was instinctively grasped and *used* (though still *unnoticed*) every time the man or animal successfully dodged or caught the coconut or stone. In these cases, common sense does not question that the properties and relations in question were really there, capable of being noticed and measured, though in fact they were not noticed. There is no reason to suppose that man has now noticed all such things. Thus, although lack of previous mention is good grounds to demand independent proof and to be skeptical of our thesis until such proof is convincingly offered, it is no ground at all for asserting our thesis to be false.

But, second, the distinctions and principles of elliptical geometry in the visual field *have* been noticed and correctly described by *some* scientific thinkers. There are places in the history of science where the concepts, operations, and principles of elliptical geometry were explicitly used and developed with respect to visible data, but due to prior commitments to Euclidean geometry, the distinctness of these principles was lost by being interpreted as belonging to a subdivision of Euclidean geometry. Thus ancient astronomers observed the "celestial sphere", i.e., the fixed stars and planets, and laid out in great detail the visible distances (visual angles) between them. They not only predicted what disposition of visibles (from fixed stars) would be seen at future times, but they also established formulae for the paths of certain visibles (from planets) among the fixed arrangements of the others. None of their calculations *required* other than principles of elliptical geometry, together with empirical data for predictions. No scientific estimate was available, or needed, as to the distance in the third dimension (away from the observer) of these bodies. There was no ground, except conjecture and an impulse to project earthbound Euclidean relations, for assuming that the fixed stars were embedded in a spherical body or any other three-dimensional shape. Thus, elliptical geometry of the visual field was correctly described by early astronomers but was dressed in Euclidean guise. The clarification of our concepts and thesis is as much or more a matter of getting rid of extraneous concepts and assumptions as of becoming aware of new things.

Still the question arises why ordinary people, if they have constantly had present to them, and have constantly been *using*, visual entities which are describable and related by principles of elliptical geometry, rarely notice the visibles and have never consciously recognized the principles which they have always been *using* in some sense subconsciously. Assuming for the moment that this question is intelligible to educated common sense, there is, I think, a plausible common sense explanation. The distinctions men notice and attend to consciously first are the distinctions men have learned that they *must* attend to consciously in order to survive or achieve their wants. Now, to survive and satisfy the wants necessary to survival, man must move about in physical space, travel from this place to that, pick up this physical object and avoid that, and make judgments on physical distances and shapes, and physical velocities. What is important for survival is not consciously to notice and distinguish the many different visual appearances and their relationships, but to notice and attend to the *physical* relationships necessary to get the apple into the mouth, or to drive an automobile, so we can live. Visibles are important to us in the sense that if men did not *subconsciously* use them properly to get hold of or avoid physical objects, mankind would probably have become extinct. But these visibles rarely require conscious attention. Visibles are not things that we eat, nor are they things which can harm us. It is not visibles as such which we must gain or avoid to survive; it is physical objects. It may or may not be considered regrettable that man's intellectual progress begins with attention to distinctions necessary for physical survival, but I think educated common sense will not deny that the broad history of man's intellectual and scientific development has indeed proceeded in this way.

Only as man's fundamental mastery and control of his physical necessities becomes secure has it been possible for man to attend more and more to questions of truth or falsehood which are not essential to his survival. There is therefore, I think, a very plausible explanation of why the conscious and explicit development of Euclidean geometry has preceded the conscious and explicit attention to the purely elliptical geometry which has been exemplified all along in the visibles present to man. The hold of Euclidean geometry, which through its marriage with classical physics has enormously increased man's ability to control and utilize physical entities for his ends, is plausibly explained in terms of the very fundamental pragmatic values which man correctly associated with Euclidean concepts and principles. What should be surprising perhaps, in the context of these considerations, is not the *late* discovery of our thesis, but rather the fact that two hundred years ago Reid, through an interest in truth alone and in the absence of any other end or pragmatic reason or precedent, should have discovered these principles so *early*.

Let us now turn to Grünbaum's first and second questions:

(1) *How* do human beings manage to get about so easily in a Euclidean physical environment even though the geometry of visual space is presumably elliptical?
(2) *How* is man able to arrive at a rather correct apprehension of the Euclidean metric relations of his environment by the use of a physiological instrument whose deliverances are claimed to be non-Euclidean?

In a sense, the second question should precede the first, for the main reason normal men do get about *easily* in physical space is because they can arrive through vision at rather correct apprehensions of Euclidean metric relations. Blind men get about in the physical world, but they clearly do not get about as *easily* as men who can see, and it is rather doubtful that mankind could survive if all men were blind. But the second question poses no great difficulties. It is simply the problem of visual space perception as it is dealt with by psychology. Standard treatments of visual space perception in psychology list the 'visual cues' which provide the basic elements upon which our perceptions or judgments of a third dimension and of the sizes of objects and of their distances from the observer depend. The basic problem is that of correlating the entities and relationships of entities in a two-dimensional visual field—or in the overlappings or comparisons of two-dimensional images, where binocular vision or parallax is concerned—with the quite different geometrical properties and relationships of the correlated three-dimensional physical objects. The only part of the standard psychological theories of visual space perception which is affected by our thesis is that portion which assumes that the two-dimensional objects of the visual field which serve as "cues", i.e., the visibles, have the properties and relationships of two-dimensional *Euclidean* geometry. What is new is that the transformations which relate visual cues to perceptual judgments of physical spatial relations must now be reformulated as transformations from a two-dimensional *elliptical* geometry rather than from a two dimensional *Euclidean* geometry. The mathematical principles of transformation involved may or may not turn out to be more complicated than the principles now assumed to hold based on a projective or perspective geometry which retains certain non-elliptical postulates originally embedded in Euclidean geometry.[4] In any case, the second question is a question for mathematicians and psychologists to work out and is neither theoretically less coherent nor notably more difficult than standard problems of this sort. Refinements and improvements are required in the psychological theory of space perceptions, but no radically new problems are presented.

Since we may now assume that very plausible accounts can be given of *how* man uses visual cues which have properties and relations of elliptical geometry to achieve rather accurate visual *perceptions* of sizes and distances of objects in three-dimensional Euclidean space, we may also assume that these accounts will help to explain in large measure how normal human beings manage to get about *so easily* in a Euclidean physical environment (question (1)).

But there is another aspect to this question; an aspect which requires that we ask how man knows the world of physical objects at all, and in particular how he knows that it has the properties and relationships of Euclidean, not elliptical, geometry. It seems probable that man would never have gotten the idea of a world of three-dimensional physical objects by vision alone, if our thesis is true. Not only is there the old question of how he could, or why he should, infer a third dimension from data which is merely two dimensional; there is also the new question, from our thesis, of how or why he would infer the existence of objects with Euclidean properties when the objects presented to his vision, his eyes, have only elliptical properties. The answer here is an old one; it is that the *basic* data by which we know physical objects do not come from vision. Berkeley and Mach long ago insisted, and many others have since agreed, that knowledge of the third dimension and its metric are primarily dependent upon touch: upon somaesthetic, tactile, and kinesthetic data. It is clear to common sense that knowledge of physical objects and their relationships and properties are not dependent on vision *alone;* for blind people, even people blind from birth, can arrive at correct judgments of Euclidean and physical relationships of objects. Further, the supposed paradigm of physical measurement—the laying of a rigid rod in contact with the object to be measured—involves (in its primitive formulation) elements of tactile, but not necessarily of visual, experience. Even more convincing, to this writer at least, is the following account of our *normal* experiences. Consider the simple occasion of walking through a doorway. Two sorts of experiences are transpiring: there is the regular, rythmic, somaesthetic sensations and jars as the feet take one, two, three, four, . . . steps, probably punctuated by the sounds of the feet hitting the floor; and there is the changing visual field, the rectangular or trapezoidal visible associated with "the doorway" which enlarges with accelerating speed until it suddenly disappears, as we "pass through the doorway". The concepts of the physical object, the doorway, and the physical process of walking through the doorway thus involve a correlation of non-visual, tactile sequences with the sequence of transformations of visibles. It is not in keeping with "common sense" to *define* or identify physical objects in terms of correlations of tactile, kinesthetic, and visual sensory data. For, in general, common sense is quickest to call those objects "real" which it must attend to in order to survive and achieve its ends, and since it is the hurts and pains of hitting a wall or falling off a precipice or the successes dependent on walking through a doorway which must be attended to and avoided or gained to achieve our ends, common sense assigns primary "reality" to the physical objects it must conceive of and attend to rather than to the various collections of sensory entities abstracted from experiences which it *uses* (usually subconsciously) as basis for its judgments and perceptions of these objects. Nor is it mere layman's common sense which adopts this stance; the physicist and the psychologist alike would reject such an identification. In a very real sense, natural and social sciences are projections of common sense, and they retain those sound basic categories and ontological commitments which the normal surviving human preserves tenaciously against philosophers. Nevertheless, it is compatible with common sense and natural science to hold that our *knowledge* of, or *awareness* of, physical objects and their Euclidean relationships is based *primarily* on tactile or kinesthetic sensation. Long before geometry or measuring sticks appeared, man found certain regularities in the numbers of steps he had to take in moving from a given spot to a seen object; he learned how, for example, to proceed from the fact that a visual image had visual size x and grew to a visual size y in taking two normal steps (kinesthetic experience) to a rough estimate of the number of additional similar steps necessary to reach the object (to make it visually fill his field) and how big it would be relative to himself when he reached it. When these subconscious rules, constantly used from prehistoric times not only by men but presumably by seeing animals as well, are formulated precisely and compatibly with our common sense judgments and findings, the result is a Euclidean theory of the geometrical properties and relations of physical objects.

Had man been, instead of an animal dependent on locomotion to survive, another kind of organism—a huge eye, unable to move itself or feel anything kinesthetic, but nevertheless surviving (as some vegetables survive)—it seems reasonable to conjecture that man would not have developed any conception of Euclidean space of physical objects and that our visibles with their elliptical geometry would constitute the structure of man's world. On the other hand, were all men born blind, and dependent on touch and feel and counting steps to get what was needed, it seems probable that men would have had to utilize subconsciously only Euclidean geometry—at least in the world as *we* have found it.

Thus, man finds his way about in a physical environment obeying Euclidean principles primarily because of what he learns from his kinesthetic and tactile experiences and sequences; he does so easily, if he has vision, by virtue of correlations (which it is the job of the psychological theory of space perceptions to make explicit) between the non-Euclidean properties and relationships of existing visibles and the Euclidean properties and relations established primarily through kinesthetic and tactile experiences.

Grünbaum's fifth question, why men like Poincaré and Helmholtz had to retrain their *Anshauung* conceptually in a counterintuitive direction before achieving a ready pictorialization of hyperbolic geometry, may be answered by reference to the historical sequence by which non-Euclidean geometries were introduced. Geometers began with the observation that one of Euclid's postulates, the parallel postulate (or related ones), was not intuitively obvious. At first the effort was to prove this postulate from others; this failed. Then the effort was to assume its falsehood and try to derive from this assumption a contradiction. This was a strictly formal effort, not one based on, or dependent upon, any positive intuition that conflicted with the parallel postulate. These attempts failed. On purely formal grounds, then—the ground that no inconsistency was involved—non-Euclidean geometries were launched. Thus, explicit non-Euclidean geometries were introduced through the back door as it were; they began as formally possible systems without any intuitive models which were compatible with ordinary conceptions of straight lines and metric relationships. Further, hyperbolic geometry came first (Lobachevski, 1829–30; Bolyai, 1832); elliptical geometry came some twenty-five years later (Riemann, 1854). There were formal reasons for this. Hyperbolic geometry involves only the alteration of one postulate, Euclid's fifth; elliptical geometry is more radically different, involving both Euclid's fifth postulate and several other Euclidean postulates and assumptions. For various reasons, mathematicians and geometers have given more attention to hyperbolic than elliptic geometry, and the accepted plan for reorganization and expansion of the field of geometry (Klein's "Erlanger program") so as to subsume all geometries, Euclidean and non-Euclidean, under projective geometry is not really suitable for the inclusion of elliptical geometry, which is therefore often subject to minor mention or neglect in mathematical treatments.

The strain which Poincaré and Helmholtz experienced in trying to fit their intuition to non-Euclidean geometry was in part due to the fact that non-Euclidean geometry was introduced first as a formal possibility; but also it was due to the fact that hyperbolic geometry despite being formally closer to Euclid has been, from the first, farther removed from any intuitive model compatible with ordinary uses of geometrical terms than elliptic geometry. It was early recognized that if we simply identified "straight lines" with "great circles" in the Euclidean geometry of a sphere, we had a model which in all respects satisfied the axioms of elliptical geometry. The infinite saddle-shaped model of hyperbolic geometry is much less intuitive. As Reid pointed out, if the observer stood in the center of the sphere and looked out, great circles would indeed *look* straight (although this seems not to have been widely noticed, since most people assumed we look at the sphere from the outside). And now, in any case, it seems clear that there is no intuitive difficulty in, or any counter-intuitive problem with respect to, elliptical geometry, whatever it may be with respect to hyperbolic geometry. The difficulty is no longer with intuition, but with cutting through accumulated common-sense assumptions about the ubiquity of Euclidean geometry and eliminating extraneous presuppositions from our perceptions of visibles.

The final questions (3) and (6) which Grünbaum suggests, again, seem to me to pose no difficulties. The answer to question (3), how students can be taught Euclidean geometry by *visual* methods if vision is non-Euclidean, is in part that students are not *in fact* taught Euclidean geometry by purely visual methods. A very important part of the teaching of Euclidean geometry involves having flat blackboards, or pieces of paper on top of flat, three-dimensional surfaces, and then using physical rulers and compasses and pencils or chalk to make straight lines and measurements on those surfaces. The relationships found to hold are relationships not between visibles as such, but between physical objects in three-space; it is the equalities and congruences of physical entities on the blackboard or paper which seems to confirm the theorems of Euclid, not the relations between the visibles which in fact do not confirm Euclid. This fact was very clearly recognized by Reid, who wrote:

> When the geometrician draws a diagram with the most perfect accuracy—when he keeps his eye fixed upon it, while he goes through a long process of reasoning, and demonstrates the relations of the several parts of his figure—he does not consider that the visible figure presented to eye, is only the representative of a tangible figure, upon which all his attention is fixed; he does not consider that these two figures have really different properties; and that what he demonstrates to be true of the one, is not true of the other.

What is true of methods of proof and learning in Euclidean geometry is also true of the laws of perspective developed in perspective geometry (which led eventually to projective geometry). In perspective geometry, the images of real or three-dimensional Euclidean objects are treated as two-dimensional Euclidean objects. Does this imply that this branch of mathematics and its history of successful applications supports the view that the two-dimensional objects of vision are Euclidean? The answer is no. Among the early students of perspective were such artists as Leonardo da Vinci and Albrecht Dürer. The result they were interested in producing directly was the physical image on the flat surface of a three-dimensional Euclidean object, their canvas, not the visibles produced in the eye. It is true that they wished the visibles produced by their canvas (when one stands directly in front of it) to coincide with the visibles produced by the real three-dimensional objects or landscape being pictured. But what Reid said of the plane geometrician working over his plane triangles and figures inked out on a flat piece of paper in three-space applies equally to the artist trying to get a "likeness" on a canvas; both are concerned with physical entities, not visibles.

As for Grünbaum's sixth question, which assumes that school children would of course find it far easier to master and grasp Euclidean geometry than a non-Euclidean geometry, I believe that his assumption, with respect to elliptical geometry, is simply wrong. Only empirical experiments in education will establish the correct answer, but I think it highly likely that, with the new sorts of instruments we have proposed and a proper textbook, school children can be taught, in the same length of time it takes to teach Euclidean plane geometry, a system of bipolar, elliptical geometry using visual triangles, quadrilaterals, and straight lines, etc. for illustrative purposes and our instruments for establishing metric relationships of equality or difference. What a child does in learning Euclidean geometry is to look at, measure with a compass or ruler, figures on a flat piece of paper or chalkboard. Even when he talks about "two-dimensional" objects, or objects in *plane* geometry, in such a course, he is using as examples objects in three-dimensional space, for flatness is a property of objects in three-space, not in two-space, and his figures and proofs in plane geometry go awry if his surfaces are not flat. He does not, therefore, measure or compare visibles. To do that, he needs other instruments, of the sort we have mentioned. Given these instruments, he can be brought to notice metric equalities and differences which he systematically ignores in plane geometry. Thus, Grünbaum's sixth question rests on the false presupposition that two-dimensional Euclidean geometry is taught by strictly visual "methods"; when this is examined critically, the difficulty is removed.

Although our answers to Grünbaum's six questions do not prove our thesis—its proof rests on evidence like that outlined in our second section—they should, I believe, successfully neutralize and remove most if not all of the reasonable hesitations which educated persons might feel because of its novelty.

APPENDIX

In a recent article on visual geometry ([7]) published after the preceding parts of this paper were accepted, James Hopkins mentions a "paradox" and offers a solution. His "paradox" is that (1) men can only see, picture, and imagine Euclidean characteristics of figures, yet (2) science now holds that the physical world is really non-Euclidean. "It seems odd," he remarks, that we are "constrained to picture the contradictory of what we have scientific reason to think is true." His solution, briefly, is that the fact that we cannot see, picture, or imagine non-Euclidean figures doesn't prove that the phenomena of vision, picturing, and imagining are necessarily Euclidean. Rather, the visual field (hence picturing and imagining) is simply too lacking in the finer-scale discriminations needed to distinguish the non-Euclidean properties which, according to scientists, hold on an astronomical scale. There is a maximum ratio of length to width which is consistent with visibility, Hopkins points out. (If, for example, two lines were 100,000 times as long as the gap separating them—e.g., 100 meters long with a gap of only 1 millimeter—men could not see both the whole line and the gap at one time; if the lines intersected at both ends, we could not distinguish by sight whether they were straight and non-Euclidean or slightly bent and Euclidean, since the necessary discriminations would involve the same too-large ratio of 100,000 to 1.) Thus, Hopkins concludes, "phenomenal figures are no more Euclidean than non-Euclidean. So phenomenal geometry is not Euclidean. Rather, it is neutral and indeterminate" ([7]:23). And since the non-Euclidean geometry ascribed to the physical world by physicists involves much greater ratios than 100,000: 1, there is no conflict or paradox.

The trouble with Hopkins' paradox is that he has it exactly wrong. The geometry of visibles, as we have argued, is *in fact* demonstrably non-Euclidean; the visibles which normal men actually see when their eyes are open and the lights are on actually have, whether noticed or not, all of the properties and relationships of an elliptic, bi-polar geometry. And the geometry of the physical world, in contrast, is Euclidean and demonstrably so in all the regions between the astronomically large and the sub-microscopically small. (Whether the findings of physicists shall be interpreted as a finding that space is curved, deduced from an operational definition of straightness as the path of propagation of light in empty space, or as a finding merely that light is deflected from Euclidean straight lines in gravitational fields is an issue we cannot pursue here; but it seems at least dubious that it is crucial to the issue.) The real problem, which is not a paradox, is to show through a psychological theory of perception how it is that man translates non-Euclidean deliverances of vision into veridical judgments about Euclidean relationships among physical objects.

Hopkins, like Kant, Mill, Strawson, Bennett, Grünbaum, and others whom he quotes, is in good company among the many intelligent men who have been mistaken on the geometry of visibles. Though he does not take the position that visual geometry is necessarily and only Euclidean, he maintains the basically erroneous position that "we cannot form the non-Euclidean pictures of our space" on the grounds that our sight is not sufficiently perfect ([7]:27). His arguments and discussions illustrate perfectly the explanations which we have given on how intelligent men might come to make such a mistake. First, he confuses *judgments* about the geometrical properties of physical objects (*Jp*) with the actual geometrical properties of visibles (*Av*). Thus, he says, following Strawson, "*x* is a phenomenal straight line" means "*x* is the look a thing has if it looks like a straight line" and "how a thing looks is connected with how it might be judged to be" ([7]:14). That he confuses, or ignores, the data available in the field of visibles with these judgments about physical things is apparent from the way he treats a passage from an article by J. R. Lucas, which asserts, as we do, that the "geometry of our visual experience is not Euclidean," and supports this by an example like ours. Hopkins quotes from Lucas:

> Let the reader look up at the four corners of the ceiling of his room, and judge what the apparent angle at each corner is; that is, at what angle the two lines where the walls meet the ceiling appear

> to him to intersect each other. If the reader imagines sketching each corner in turn, he will soon convince himself that all the angles are more than right angles, some considerably so. And yet the ceiling appears to be a quadrilateral. From which it would seem that the geometry of appearance is non-Euclidean.

But Hopkins rejects the argument on the basis of a classic confusion which Reid pointed out two centuries ago. Picking up the suggestion to *sketch,* Hopkins says "But in no sketch will the reader draw a quadrilateral with four visibly obtuse angles. None can be drawn." ([7]:13.) The impossibility he speaks of is of course the fact; but it says nothing about an impossibility of *seeing* non-Euclidean quadrilaterals. It is a *physical* impossibility he points to; the impossibility of making a quadrilateral on a flat piece of paper in Euclidean three-space, measuring its angles by putting a protractor adjacent to its angles, and finding the angles, as measured by the protractor, to add up to more than 360°. If one *defines* "visual quadrilateral" as "what we would *judge* the quadrilateral on the piece of flat paper to be" (as he does), then no doubt most people would *judge* the physical angles to be four right angles even when they appeared as four acute or obtuse angles in a trapezoid. But if we distinguish our judgments about physical objects (*Jp*) from the actual properties of visual figures (*Av*)—as Lucas was plainly doing and Hopkins was plainly refusing to do—then it is apparent that appearances of the four angles in the ceiling are angles with a sum of more than 360°. Hopkins was guilty, in Reid's words, of failing to see that the "visible figure presented to the eye, is only the representative of the tangible figure, upon which all of his attention is fixed . . . and that what he demonstrates to be true of the one, is not true of the other."

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[12] Wolfe, Harold E., *Introduction to Non-Euclidean Geometry* (New York: Holt, Rinehart, and Winston, 1945): 66, 81, 82, and 89 for theorems 1, 2, 3, and 4, respectively.

Notes

[1] This paper is a revision of a paper entitled "Geometry and the Pure Visual Field" read at the meetings of the American Philosophical Association, Western Division, in Columbus, Ohio, on May 3, 1963. The new title is that of Section IX, Ch. VI, in Thomas Reid's *An Inquiry into the Human Mind on the Principles of Common Sense* [10] and is adopted in deference to Reid's priority, of which I was not fully aware when the first paper was written.

[2] Reid was the first, and I believe the only, philosopher of standing to hold the thesis set forth in this paper. Only recently, as in Norman Daniels' article [5], has Reid's position begun to attract the attention it deserves.

[3] Roberts and Suppes [11] also take Luneberg's data as a good basis for developing a "geometry of visual perception". They are fairly explicit in viewing these data as *judgments* about distances, perpendicularity, and parallelism among *physical objects* (our *Jp*). Yet they speak of the geometry it leads to as dealing, interchangeably, with "perceptual space", "visual space", "subjective visual space", and "primitive visual space", thus obscuring or ignoring the distinction between geometries based on relations and properties of *judgments* about *physical objects* (*Jp*) and geometries based on relations and properties found among *actual visibles* (*Av*) themselves.

[4] Projective geometry assumes for example that two points always determine one and only one straight line; this is true in Euclidean but not in Elliptical geometry.